HELICAL FLOWS IN A SPHERE

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UDC 532.5

Let Ω be a domain in R^3 containing the origin and bounded by the smooth surface S with unit external normal vector **n**. The vector field **v**, smooth in the class C^1 in Ω and continuous in $\bar{\Omega}$ for which

$$\mathrm{rot}\,\mathbf{v} = \lambda \mathbf{v}; \tag{1}$$

$$\mathbf{v}\cdot\mathbf{n}|_S = 0 \tag{2}$$

($\lambda = \mathrm{const} \neq 0$), is called a homogeneous helical flow. From (1) we obtain div $\mathbf{v} = 0$, hence, rot rot $\mathbf{v} = -\Delta\mathbf{v} = \lambda^2\mathbf{v}$, and the Cartesian components of the field **v** should be analytic functions in Ω. In particular, **v** decomposes into the Taylor series

$$\mathbf{v} = \sum_{p=0}^{\infty} \mathbf{v}_p, \tag{3}$$

which converges uniformly in some neighborhood of the origin. Here each $\mathbf{v}_p$ is a homogeneous polynomial vector field of degree p and divergence zero. The following lemmas describe the connection of the fields $\mathbf{v}_p$ to the harmonic polynomials:

LEMMA 1. If H_n is a homogeneous harmonic polynomial of degree n, then there exists a sequence $\{\mathbf{v}_{s,n}\}$ ($s = 0, 1, 2\ldots$) of homogeneous polynomial fields ($\deg \mathbf{v}_{s,n} = n + s - 1$) satisfying the conditions

$$\mathbf{v}_{0,n} = \mathrm{grad}\, H_n; \tag{4}$$

$$\mathrm{rot}\, \mathbf{v}_{s,n} = \mathbf{v}_{s-1,n} \quad \text{as} \quad s \geqslant 1. \tag{5}$$

Proof. Let us give $\mathbf{v}_{s,n}$ by the formulas

$$\mathbf{v}_{2k,n} = C_{kn} r^{2k-2}[(n+2k+1)r^2\, \mathrm{grad}\, H_n - 2knH_n\mathbf{r}]; \tag{6}$$

$$\mathbf{v}_{2k+1,n} = C_{kn} r^{2k}\, \mathrm{grad}\, H_n \times \mathbf{r}, \tag{7}$$

where

$$C_{kn} = \frac{\Gamma\left(n+\frac{3}{2}\right)}{n+1}\frac{(-1)^k}{2^{2k}k!\,\Gamma\left(n+k+\frac{3}{2}\right)}; \quad \mathbf{r} = \{x, y, z\}; \quad r^2 = x^2 + y^2 + z^2. \tag{8}$$

The property (4) is evident. The property (5) is verified by using standard vector analysis formulas and the formula

$$\mathrm{rot}\,(\mathbf{v}_p \times \mathbf{r}) = (p+2)\mathbf{v}_p - \mathbf{r}\,\mathrm{div}\,\mathbf{v}_p,$$

which is valid for a homogeneous vector field $\mathbf{v}_p$ of degree p and obtained from the formula

$$\mathrm{rot}\,(\mathbf{v}_p \times \mathbf{r}) = (\mathbf{r}, \nabla)\mathbf{v}_p - (\mathbf{v}_p, \nabla)\mathbf{r} + \mathbf{v}_p\,\mathrm{div}\,\mathbf{r} - \mathbf{r}\,\mathrm{div}\,\mathbf{v}_p,$$

taking into account that $(\mathbf{r}, \nabla)\mathbf{v}_p = p\mathbf{v}_p$ (the Euler theorem about homogeneous functions), $(\mathbf{v}_p, \nabla)\mathbf{r} = \mathbf{v}_p$, div $\mathbf{r} = 3$.

LEMMA 2. For any field **v** satisfying (1) and having the Taylor expansion (3), there exists a sequence $\{H_n\}$ ($n = 1, 2, 3, \ldots$) of homogeneous harmonic polynomials of deg $H_n = n$, such that for all p

$$\mathbf{v}_p = \sum_{n=1}^{p+1} \lambda^{p-n+1}\mathbf{v}_{p-n+1,n}, \tag{9}$$

where $\mathbf{v}_{p-n+1,n}$ is the field constructed by means of H_n because of Lemma 1.

Proof. For $p = 0$ we have $\mathbf{v}_0 = \mathrm{grad}\, H_1 = \mathbf{v}_{0,1}$, where the linear form H_1 is defined uniquely. Let H_n be defined for $1 \le n \le p+1$ and let (9) hold. Let us define H_{p+2} so that the equality (9) would be satisfied by replacing p by p+1. Because of the uniqueness of the Taylor expansion we have

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Chelyabinsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, March–April, 1979. Original article submitted March 29, 1978.

 0021-8944/79/2002-0164$07.50

$$\operatorname{rot} \mathbf{v}_{p+1} = \lambda \mathbf{v}_p = \sum_{n=1}^{p+1} \lambda^{p-n+2} \mathbf{v}_{p-n+1,n}.$$

Then according to Lemma 1

$$\operatorname{rot}\left(\mathbf{v}_{p+1} - \sum_{n=1}^{p+1} \lambda^{p-n+2} \mathbf{v}_{p-n+2,n}\right) = \lambda \mathbf{v}_p - \lambda \mathbf{v}_p = 0,$$

hence $\mathbf{v}_{p+1} - \sum_{n=1}^{p+1} \lambda^{p-n+2} \mathbf{v}_{p-n+2,n} = \operatorname{grad} H_{p+2}$, where the polynomial H_{p+2} is defined uniquely by the requirement of homogeneity. We have

$$\Delta H_{p+2} = \operatorname{div}\left(\mathbf{v}_{p+1} - \sum_{n=1}^{p+1} \lambda^{p-n+2} \mathbf{v}_{p-n+2,n}\right) = 0,$$

hence, the polynomial H_{p+2} is harmonic, and we can set $\mathbf{v}_{0,p+2} = \operatorname{grad} H_{p+2}$, from which $\mathbf{v}_{p+1} = \sum_{n=1}^{p+2} \lambda^{p-n+2} \mathbf{v}_{p-n+2,n}$.

The lemma is proved.

For each n let us define the vector field

$$\mathbf{v}^{(n)} = \sum_{s=0}^{\infty} \lambda^s \mathbf{v}_{s,n}, \tag{10}$$

where $\mathbf{v}_{s,n}$ is the field constructed in Lemma 1 by means of the harmonic polynomial H_n from Lemma 2. Formula (8) for C_{kn} shows that the series (10) converges uniformly for all **r**. This series can be differentiated term by term, and hence $\operatorname{rot} \mathbf{v}^{(n)} = \lambda \mathbf{v}^{(n)}$, and $\mathbf{v}^{(n)}$ can be evaluated explicitly. To do this, we note that $\mathbf{v}^{(n)} = \mathbf{v}_+^{(n)} + \mathbf{v}_-^{(n)}$, where the components are the "even" and "odd" parts of $\mathbf{v}^{(n)}$ corresponding to (6) and (7):

$$\mathbf{v}_+^{(n)} = \sum_{k=0}^{\infty} \lambda^{2k} \mathbf{v}_{2k,n}, \quad \mathbf{v}_-^{(n)} = \sum_{k=0}^{\infty} \lambda^{2k+1} \mathbf{v}_{2k+1,n},$$

where $\mathbf{v}_+^{(n)} = \lambda^{-1} \operatorname{rot} \mathbf{v}_-^{(n)}$ so that it is sufficient to find $\mathbf{v}_-^{(n)}$. It follows from (7) and (8) that

$$\mathbf{v}_-^{(n)} = \frac{\lambda \Gamma\left(n+\frac{3}{2}\right)}{n+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma\left(n+k+\frac{3}{2}\right)} \left(\frac{\lambda r}{2}\right)^{2k} \operatorname{grad} H_n \times \mathbf{r} = A_n r^{-n-1/2} J_{n+1/2}(\lambda r) \operatorname{grad} H_n \times \mathbf{r}, \tag{11}$$

where $A_n = \lambda(n+1)^{-1}\Gamma(n+3/2)$; $J_{n+1/2}(\lambda r)$ is the Bessel function.

Let $S_r \subset \Omega$ be a sphere of radius r with center at the origin. Let us show that

$$\int_{S_r} \mathbf{v} \cdot \mathbf{r} H_n dS = \int_{S_r} \mathbf{v}^{(n)} \cdot \mathbf{r} H_n dS. \tag{12}$$

To do this we examine the partial sum S_m of the series (3). By virtue of Lemma 2

$$\mathbf{S}_m = \sum_{p=0}^{m} \mathbf{v}_p = \sum_{p=0}^{m} \sum_{n=1}^{p+1} \lambda^{p-n+1} \mathbf{v}_{p-n+1,n} = \sum_{n=1}^{m+1} \sum_{p=n-1}^{m} \lambda^{p-n+1} \mathbf{v}_{p-n+1,n} = \sum_{n=1}^{m+1} \sum_{s=0}^{m-n+1} \lambda^s \mathbf{v}_{s,n} = \sum_{n=1}^{m+1} \mathbf{S}_{m-n+1}^{(n)},$$

where $S_{m-n+1}^{(n)}$ is the partial sum of the series (10). We have

$$\mathbf{S}_m \cdot \mathbf{r} = \sum_{n=1}^{m+1} \mathbf{S}_{m-n+1}^{(n)} \cdot \mathbf{r} = \sum_{n=1}^{m+1} f_{m-n+1}(r) H_n,$$

where $f_{m-n+1}(r)$ are polynomials whose explicit form can be extracted from (6). From the last equality and the orthogonality of the spherical functions of different orders there follows that for $m \geq n-1$

$$\int_{S_r} \mathbf{S}_m \cdot \mathbf{r} H_n ds = f_{m-n+1}(r) \int_{S_r} H_n^2 ds = \int_{S_r} \mathbf{S}_{m-n+1}^{(n)} \cdot \mathbf{r} H_n dS. \tag{13}$$

Let $r < \varepsilon$, where ε is the radius of a sphere in which the series (3) converges uniformly. Then it is possible to pass to the limit as $m \to \infty$ under the integral sign in both sides of (13), and (12) is proved for $r < \varepsilon$. Since both sides of this equality are analytic functions of r, then it is true for all r for which $S_r \subset \Omega$.

Now, let Ω be a sphere of radius R with center at the origin. By continuity, (12) is satisfied even for $r = R$, where it follows from (2) that both its sides equal zero. Let n be the first number for which $H_n \neq 0$. We have

$$\mathbf{v}^{(n)}\cdot\mathbf{r} = \left(\mathbf{v}_+^{(n)} + \mathbf{v}_-^{(n)}\right)\cdot\mathbf{r} = \mathbf{v}_+^{(n)}\cdot\mathbf{r},$$

since $\mathbf{v}_-^{(n)}\cdot\mathbf{r} = 0$ by virtue of (11). Furthermore, $\mathbf{v}_+^{(n)}\cdot\mathbf{r} = \lambda^{-1}\,\mathrm{rot}\,\mathbf{v}_-^{(n)}\cdot\mathbf{r} = \lambda^{-1}\,\mathrm{div}(\mathbf{v}_-^{(n)}\times\mathbf{r})$, from which we obtain by using (11)

$$\mathbf{v}_+^{(n)}\cdot\mathbf{r} = n\Gamma\left(n+\frac{3}{2}\right) r^{-n-1/2} J_{n+1/2}(\lambda r)\, H_n.$$

The equality (12) becomes

$$n\Gamma\left(n+\frac{3}{2}\right) R^{-n-1/2} J_{n+1/2}(\lambda R) \int\limits_S H_n^2\, ds = 0, \tag{14}$$

from which $J_{n+1/2}(\lambda R) = 0$ and $\lambda = \mu_k^{(n)} R^{-1}$, where $\mu_k^{(n)}$ is the k-th root of the function $J_{n+1/2}(z)$.

Now let $m > n$. Since $J_{n+1/2}(z)$ and $J_{m+1/2}(z)$ have no common zeroes, then from an equality analogous to (14) with n replaced by m, we obtain $\int\limits_S H_m^2 ds = 0$, from which $H_m = 0$ for all $m \neq n$. Summarizing, $\mathbf{v} = \mathbf{v}^{(n)}$, and the following theorem is proved:

THEOREM. Every homogeneous helical flow in a sphere of radius R has the form $\mathbf{v} = \mathbf{v}_+^{(n)} + \mathbf{v}_-^{(n)}$, where $\mathbf{v}_-^{(n)}$ is given by (11) for $\lambda = \mu_k^{(n)} R^{-1}$, $\mathbf{v}_+^{(n)} = [\mu_k^{(n)}]^{-1} R\,\mathrm{rot}\,\mathbf{v}_-^{(n)}$, and H_n is an arbitrary homogeneous harmonic polynomial of degree n. In particular, $2n+1$ linearly independent helical flows correspond to each value of $\lambda = \mu_k^{(n)} R^{-1}$.

Remark 1. When H_n possesses axial symmetry, the solutions obtained are known (see [1], for instance). If $n = 1$, then symmetry relative to the axis given by the vector grad H_1 automatically holds. The streamline pattern in the meridian section is represented in [2] for $n = k = 1$.

Remark 2. Let

$$H_n^m = r^n P_n^m(\cos\theta)\cos m\varphi \quad \text{at} \quad 0 \leqslant m \leqslant n,$$
$$H_n^m = r^n P_n^{|m|}(\cos\theta)\sin|m|\varphi \quad \text{at} \quad -n \leqslant m < 0.$$

It can be shown that the family of vector fields $\{\mathbf{v}_+^{(n)}, \mathbf{v}_-^{(n)}\}$ for all possible n, $k \geq 1$, $|m| \leq n$, $\lambda = \mu_k^{(n)} R^{-1}$, $H_n = H_n^m$ is the orthogonal basis in the space $J^0(\Omega)$ [3]. The proper basis of the operator $\tilde{\Delta}$ from [3] in the sphere Ω is also associated with the fields $\mathbf{v}_+^{(n)}$ and $\mathbf{v}_-^{(n)}$. It is formed by the fields $\mathbf{v}_-^{(n)}$ for all possible n, k, m, $\lambda = \mu_k^{(n)} R^{-1}$ and the fields

$$\mathbf{v}_*^{(n)} = \mathbf{v}_+^{(n)} - \Gamma\left(n+\frac{3}{2}\right) R^{-n-1/2} J_{n+1/2}(\mu_k^{(n+1)})\,\mathrm{grad}\, H_n^m$$

for all possible n, k, m, $\lambda = \mu_k^{(n+1)} R^{-1}$. Here $\mathbf{v}_-^{(n)}$ corresponds to the eigenvalue $\nu[\mu_k^{(n)} R^{-1}]^2$, and $\mathbf{v}_*^{(n)}$ to the eigenvalue $\nu[\mu_k^{(n+1)} R^{-1}]^2$. This basis affords the possibility of rapidly solving the Cauchy problem for a system of viscous fluid motion equations in a sphere in a Stokes linearization by using the Fourier method.

LITERATURE CITED

1. A. G. Yarmitskii, "Sphere in a homogeneous helical flow," in: Hydroacromechanics and the Theory of Elasticity [in Russian], No. 19, Kharkov (1975).
2. A. G. Yarmitskii, "On a three-dimensional analog of the Chaplygin vortex column," Prikl. Mekh. Tekh. Fiz., No. 5 (1974).
3. O. A. Ladyzhenskaya, Mathematical Questions of the Dynamics of a Viscous Incompressible Fluid [in Russian], Nauka, Moscow (1970).